# 7) From local to global



Extension of special coordinates and critical points.

In the previous section, given a holomorphic germ $f:(\mathbb{C},0)\circlearrowleft$, we constructed special coordinates $\Phi$ that conjugate $f$ to their normal form (either on a nbhd of $0$ or on some subdomain):

- $0<|\lambda|<1$ (attracting case) $\rightsquigarrow$ Koenigs coordinates: $\Phi\circ f=\lambda\Phi$.
  unique until $\Phi'(0)=1$
- $\lambda=0$ (superattracting case) $\rightarrow$ Böttcher coordinates: $\Phi\circ f=\Phi^d$ (unique up to $(d-1)$-roots of $1$)
- $\lambda=e^{2\pi i\frac{p}{q}}$ (parabolic) $\rightsquigarrow$ Fatou coordinates: $\Phi\circ f(z)=\Phi(z)+1$ (on petals, unique up to translations)
- $\lambda=e^{2\pi i\alpha}$ (irrational): if $f$ is linearizable $\rightsquigarrow$ Siegel coordinates: $\Phi\circ f=\lambda\Phi$ (unique until $\Phi'(0)=1$)

We now consider a rational map $f:\hat{\mathbb{C}}\circlearrowleft$, a fixed point $p\in\hat{\mathbb{C}}$, and the associated germ $f_p:(\hat{\mathbb{C}},p)\circlearrowleft$. We study the ~~max~~ largest domain of definition of the special coordinates.

## I. Attracting case.

**Lemma:** $f:X\circlearrowleft$, $p\in X$ attracting fixed point. Then the Koenigs coordinates (unique up to mult. by const.) $\Phi:(X,p)\to(\mathbb{C},0)$ extends to a holomorphic map $\Phi:\mathcal{A}\to\mathbb{C}$, where $\mathcal{A}$ is the basin of attraction to $p$.

*Proof:* Let $\Phi_0:U\to\mathbb{C}$ be the Koenigs coordinate, defined on a nbhd $U$ of $p$. Then we set $\Phi(z)=\lambda^{-n}\left(\Phi_0\circ f^n(z)\right)$, where $n$ is big enough so that $f^n(z)\in U$. $\square$

7.2

The map $\Phi$ won't be injective in general. The lack of injectivity will be given by the presence of critical points.

To state the result, notice that locally $\Phi$ admits an inverse $\Psi = \Phi^{-1}$, defined on some small disk. It extends to some maximal open disk $\mathbb{D}_r$.

$f : \hat{\mathbb{C}} \circlearrowleft$ of degree $d \geq 2$, $\Phi$ Koenigs coordinate of the attr. fixed point $p$. $\Psi$ its local inverse.

Lemma: $\Psi$ extends homeomorphically on $\partial \mathbb{D}_r$, and $\Psi(\partial \mathbb{D}_r) \subset \mathcal{A}_0$ contains a critical point for $f$.

Proof: We try to extend $\Psi$ radially on $R_\theta = \{ \rho e^{2\pi i \theta} \mid \rho \geq 0 \}$. $\forall \theta \in \mathbb{R}/\mathbb{Z}$.

This is not possible to do indefinitely (i.e. $\rho \to \infty$) for any $\theta$.

In fact, if so, we have $\Psi : \mathbb{C} \to \mathcal{A}_0 \subset \hat{\mathbb{C}}$, with $\Phi \circ \Psi = id$.

This would give $\Psi(\mathbb{C})$ a simply connected parabolic Riemann surface in $\hat{\mathbb{C}}$, thus $\Psi(\mathbb{C}) = \hat{\mathbb{C}} \setminus \{q\}$ for some point $q$.

This would imply that $f = \Psi \circ \underset{\substack{\shortparallel \\ z \mapsto \lambda z}}{f_*} \circ \Phi$ $(|\hat{\mathbb{C}} \setminus q|)$ is 1-to-1, against the hypothesis $d \geq 2$.

$\Rightarrow \exists r$ some largest radius, for which $\Psi$ extend to $\Psi : \mathbb{D}_r \to \mathcal{A}_0$

Set $U = \Psi(\mathbb{D}_r)$.

Notice that $\overline{U} \subset \mathcal{A}_0$! In fact, since $\tilde{f}(\overline{\mathbb{D}_r}) = \lambda \overline{\mathbb{D}_r} = \overline{\mathbb{D}}_{|\lambda| r} \subset\subset \mathbb{D}_r$,

the image $\overset{f(\overline{U})}{}$ $f(\overline{U})$ is a compact subset $K \subset\subset U$ cont.

Being $\mathcal{A}$ $f$-invariant, we get $\overline{U} \subset \mathcal{A}$.

In particular, $\Phi$ is defined and holomorphic in a nbhd of $\overline{U}$.

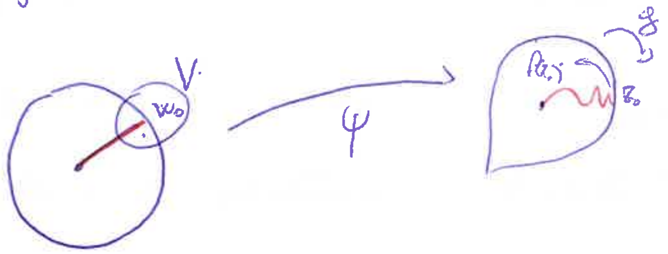
We now show that $\partial U \cap C_f \neq \emptyset$, or else we could extend $\varphi$ to a larger disk $\mathbb{D}_{r'}$, $r'>r$, against the maximality of $r$.

Pick any point $w_0 \in \partial \mathbb{D}_r$, consider the curve $\overset{[0,1)}{t} \mapsto \varphi(t w_0)$, and pick an accumulation point $z_0$ (for $t \to 1$).

If $z_0$ is not a critical point for $f$, we can choose a holomorphic branch $g$ of $f^{-1}$ in some neighborhood of $f(z_0)$, where we have $g(f(z_0)) = z_0$.

Hence we can extend $\varphi$ holomorphically on some nbhd $V$ of $w_0$, by:

$$\varphi(w) = g(\varphi(\lambda w)).$$




If $\partial U \cap C_f = \emptyset$, these extensions would patch together to give a $\varphi : \mathbb{D}_{r'} \to A_0$, $r' > r$.

We finally show that $\Phi$ maps $\overline{U}$ to $\overline{\mathbb{D}}_r$ homeomorphically.

it suffices to show that $\Phi$ is injective on $\partial U$: $\forall z \neq z' \in \partial U \Rightarrow \Phi(z) \neq \Phi(z')$

Suppose the contrary, and $\Phi(z) = \Phi(z') =: w \in \partial \mathbb{D}_r$.

Pick sequences $z_j \to z$ and $z_j' \to z'$ in $U$, so that $\Phi(z_j)$, $\Phi(z_j')$ converge to the same limit $w$. Let $I_j = [\Phi(z_j), \Phi(z_j')]$, and let $\Omega$ be the set of accumulation points for $\varphi(I_j)$ as $j \to \infty$.

Then $\Omega$ is compact, connected and contains $z, z'$.

Contains $z, z'$ by construction.

$\Omega = \bigcap_{n \in \mathbb{N}} \overline{\bigcup_{m \geq n} \varphi(Q_m)}$, intersection of nested connected sets.

$Q_m$: quadrilateral between $I_m$ and $I_{m+1}$. In fact $\bigcup_{m \geq n} Q_m$ is a compact connected set in $\overline{\mathbb{D}}_r$ if we add $w$.

Hence $\Omega$ is compact and connected.

But $\forall \zeta \in \Omega$, $f(\zeta) = \lim f(\varphi(\underset{\in I_j}{w_j})) = \lim \varphi(\lambda \underset{\downarrow w}{w_j}) = \varphi(\lambda w)$,

which gives a contradiction ($f^{-1}(p)$ is finite). □

Consider more generally an attracting cycle $z_0, z_1 \dots z_{m-1}$.

Its immediate basin of attraction is given by $\mathcal{A}_0 = \bigcup_{j=0}^{m-1} \mathcal{A}_0(z_j, f^m)$

Thm (Fatou, Julia). Let $f: \hat{\mathbb{C}} \circlearrowleft$ be a rational map of degree $d \geq 2$.

Then the immediate basin of every contracting periodic orbit contains at least a critical point.

In particular, the number of contracting periodic orbits is finite, bounded by the number of critical points

Proof: notice that a superattracting germ of order $p$ contributes with multiplicity $p-1$.

if the cycle is superattracting, one of the $z_j$ is the critical point in $\mathcal{A}_0$.

If the cycle is attracting of length 1, this is given by the previous result.

If $z_0 \dots z_{m-1}$ is a (attracting) cycle, then $f(\mathcal{A}_0(z_j, f^m)) \subset \mathcal{A}_0(z_{j+1}, f^m)$.

If none of the $\mathcal{A}_0(z_j, f^m)$ contains a critical point for $f$, then ~~$\mathcal{A}_0(z_0, f^m)$~~ by chain rule, they do not contain a critical point for $f^m$, which is a contradiction.

$\square$

Rem: $\# \mathcal{C}_f = 2d-2$ (counted with multiplicity).

(Direct computation: up to change of coordinates on the target space, we may assume $f(\infty) = \infty$ $\Leftrightarrow \underset{n}{\deg P} > \underset{m}{\deg Q}$ and we get that $\infty$ is critical of multiplicity $n-m-1$, while $f' = 0 \Leftrightarrow P'Q - PQ' = 0$ which has $n+m-1$ solutions. $\sum = 2n-2 = 2d-2$

Rem: no analogous in higher dimensions.

Theorem (topology of $\mathcal{A}_0$). Let $\mathcal{A}_0$ be the immediate basin of attraction of a contracting fixed point (of $f: \hat{\mathbb{C}} \circlearrowleft$, $\deg f \geq 2$).

Then $\hat{\mathbb{C}} \setminus \mathcal{A}_0$ is either connected or has $\infty$-many connected components.

Rem: equivalently, $\mathcal{A}_0$ is either simply-connected or $\infty$-connected.

Proof. Pick a small open disk $N_0 = B(z_0, \varepsilon)$, where $z_0$ is the fixed point, $\varepsilon \ll 1$, and such that $\partial N_0 \cap PC(f) = \emptyset$ (i.e., no forward orbit of critical points belong to $\partial N_0$).

$PC$ ^ postcritical

We set $N_k$ equal to the connected component of $f^{-k}(N_0)$ containing $z_0$.

Notice that $N_0 \subset N_1 \subset N_2 \ldots$ and $\bigcup_{k=0}^{\infty} N_k = \mathcal{A}_0$

In fact, if $z \in \mathcal{A}_0$, take a path $\gamma \subset \mathcal{A}_0$ joining $z$ and $z_0$.
By compactness, $\exists k$, $f^k(\gamma) \subset N_0 \Rightarrow \gamma \subset N_k$.

Notice that since since $\partial N_0 \cap PC(f) = \emptyset$, $f^{-k}(\partial N_0)$ is a disjoint union of topological circles. It follows that $\partial N_k$ is a finite number of (topological) topological circles, and $\hat{\mathbb{C}} \setminus N_k$ is the disjoint union of finitely many discs (bounded by the same circles)

There are now two possibilities:

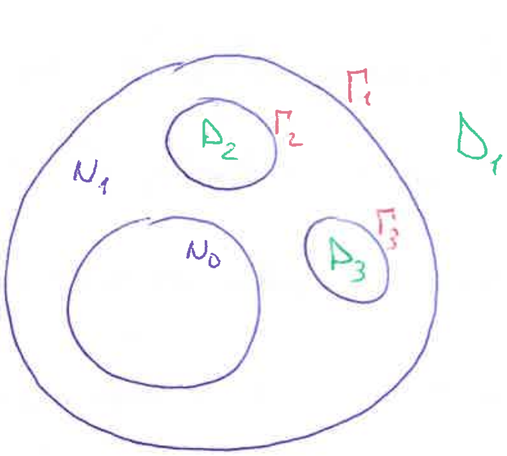
Case 1: $\forall k$, $N_k$ is bounded by only one circle: $\hat{\mathbb{C}} \setminus N_k$ is a topological disc. (in particular, connected) and $\hat{\mathbb{C}} \setminus \mathcal{A}_0 = \bigcap_{k=0}^{\infty} \hat{\mathbb{C}} \setminus N_k$ is connected

Case 2: Suppose this is not the case

Then there is a smallest integer ~~k~~ $m$ so that $\partial N_m$ is not connected.

~~Up to replace $N_0$ by~~ Up to rename $N_{m-1}$ as $N_0$, we may assume $m=1$.

Call $\Gamma_1 \dots \Gamma_n$ the connected components of the boundary of $N_1$, $(n \geq 2)$. $\Gamma_j$ bounds a disk $D_j$ in $\hat{\mathbb{C}} \setminus N_1$.




We will show that for any $(i_1 \dots i_k) \in \{1, \dots, n\}^k$, $\partial N_k$ has at least a connected component $\Gamma_{i_1 \dots i_k}$ in $D_{i_1}$ so that $f(\Gamma_{i_1 \dots i_k}) = \Gamma_{i_2 \dots i_k}$.

(We construct such $\Gamma_{i_1 \dots i_k}$ inductively). The base of the induction is exactly our starting construction of $\Gamma_1 \dots \Gamma_n$.

Since $f : \hat{\mathbb{C}} \circlearrowleft$ is a branched covering, and $f(N_k) = N_{k-1}$, $f$ defines a branched covering $\overline{N_k} \to \overline{N_{k-1}}$;

In particular, we have a branched covering $\overline{N_k} \setminus f^{-1}(N_0) \to \overline{N_{k-1}} \setminus N_0$ and the same holds for any connected component $U$ of $\overline{N_k} \setminus f^{-1}(N_0)$.

Notice that $U \subset D_i$ for some $i$, and that for any $i = 1 \dots n$, there is one such $U$, $U \subset D_i$. The first part is given by the fact that $\overline{N_k} \setminus f^{-1}(N_0) \subset \overline{N_k} \setminus N_1 \subset \hat{\mathbb{C}} \setminus N_1$. The second part is given by the fact that $N_k$ is a nbhd of $N_{k-1}$ for any $k$.

In particular, any of the curves $\Gamma_{i_2 \dots i_k}$ in $\partial N_{k-1}$ is covered by at least a curve $\Gamma_{i_1, i_2 \dots i_k} \subset D_{i_1} \cap \partial N_k$.

It follows that $D_{i_1 \dots i_k} \subset \hat{\mathbb{C}} \setminus N_k$ are all disjoint, and $D_{i_1} \supset D_{i_1 i_2} \supset \dots$

Thus $\hat{\mathbb{C}} \setminus \mathcal{A}_0$ contains a component $\bigcap_{k=0}^{\infty} \overline{D}_{(i_1 \dots i_k)} \neq \emptyset$ for any choice of infinite sequence $(i_k)_k \in \{1 \dots n\}^{\mathbb{N}}$. Hence $\hat{\mathbb{C}} \setminus \mathcal{A}_0$ has uncountably many connected components.

□